Lecture 7: More on $\mathrm{SU}(3)_{\mathrm{flavor}}$

- Review from Last Time
- SU(2) and SU(3): Group Theory and Quark Model Interpretation
- The Meson Multiplets
- The Baryon Multiplets
- Antiparticles
- Fermi Statistics and the Need for Color?
- SU(3) Breaking and Mass Formulaes
- Magnetic Moments

Review From Last Time

- Strongly interacting particles (hadrons) classified as spin-1/2 baryons and integer spin mesons
  - $\exists$ families of particles with same spin and parity and similar masses, but with different charges.
  - Strong interactions of particles within a family are the same
  - Eg, protons and neutrons have very similar masses and see same nuclear interactions
- Postulate that particles within a family are related by a symmetry property
  - Isospin
  - Once strangeness discovered, extend to SU(3)

Charge: Determined from Strangeness and I_Z

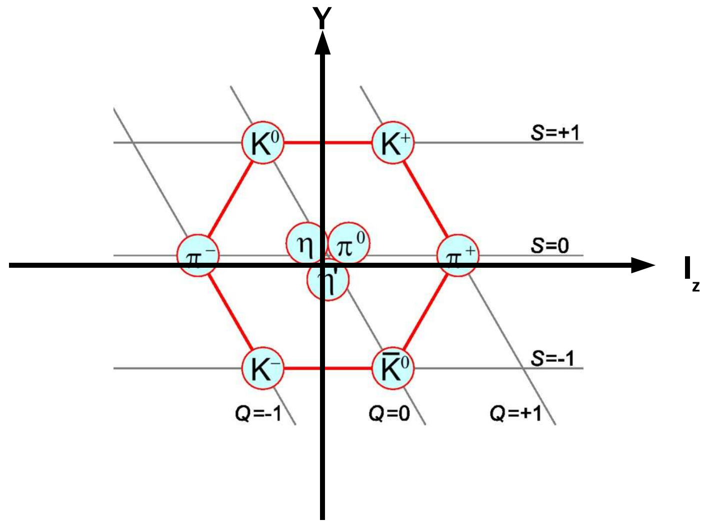
- We've already seen that within an isospin multiplet, different I_z have different charge
- Can generalize this observation for all light quark (u,d,s) multiplets:

$$Q = I_z + \frac{B+S}{2}$$

  Define hypercharge $Y \equiv B + S$

- This is called the Gell Mann-Nishijima Eq
- Note: Because Q is determined from I_3, EM interactions cannot conserve isospin, but do conserve I_3
  - This is analogous to the Zeeman effect in atomic physics where a B field in z direction destroys conservation of angular momentum, but leaves J_z as a good quantum number
- EM coupling $\sim\ 1\%$ so effects of isospin non-conservation are small and can be treated as perturbative correction to strong interaction

Group Theory Interpretation



- Describe particles with same spin, parity and charge congugation symmetry as members of a multiplet with different I_z and Y
- Will define (next 2 pages) raising and lowering operators to navigate around the multiplet
- Gell Man and Zweig suggested that patterns of multiplets could be explained if all hadrons were made of quarks
  - Mesons: $q\overline{q}$ $\quad 3 \otimes \overline{3} = 1 \oplus 8$
  - Baryons: qqq $\quad 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$
- In those days, 3 flavors (extension to 6 discussed later)

Introduction to Group Theory (via SU(2))

- Let's start by reviewing SU(2) Isospin
- Fundamental representation: a doublet

$$\chi = \begin{pmatrix} u \\ d \end{pmatrix} \quad \text{so} \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Define rotation in isospin space in terms of infinitesimal generators of the rotations

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The τ matrices satisfy commutation relations

$$[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j] = i\,\frac{1}{2}\epsilon_{ijk}\tau_k$$

  These commutation relations define the SU(2) algebra

- We can have higher representations of SU(2): $N \times N$ matrices with $N = 2I + 1$
- Also, there is an operator that commutes with all the τ's:

$$I^2 = (\frac{1}{2}\vec{\tau})^2 = \frac{1}{4}\Sigma_i\ \tau_i^2$$

  and there are raising and lowering operators

$$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

Extension to SU(3))

- SU(3): All unitary transformations on 3 component complex vectors without the overall phase rotation (U(1))

$$U^\dagger U = UU^\dagger = 1 \qquad \det U = 1$$
$$U = exp[i \sum_{a=1}^{8} \lambda_a \theta_a / 2]$$

- The fundamental representation of SU(3) are 3×3 matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Commutation relations:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

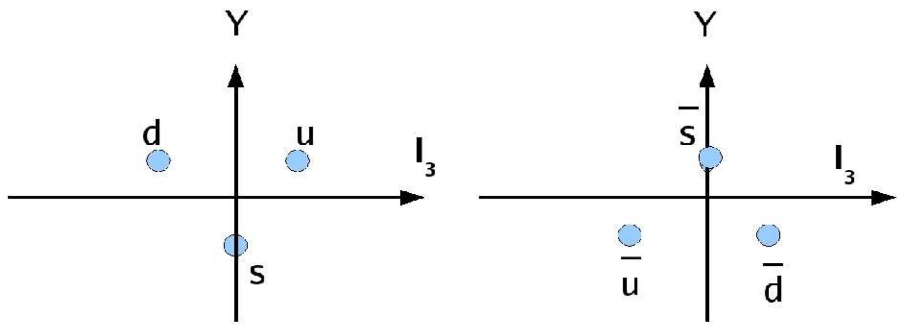
where $f_{123} = 1$, $f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$, $f_{156} = f_{367} = -\frac{1}{2}$ and $f_{458} = f_{678} = \sqrt{3}/2$.

SU(3) Raising and Lowering Operators

- SU(3) contains 3 SU(2) subgroups embedded in it

$$\begin{array}{llll} \text{Isospin}: & F_1 & F_2 & F_3 \\ \text{U}-\text{spin}: & F_6 & F_7 & \sqrt{3}F_8 - F_3 \\ \text{V}-\text{spin}: & F_4 & F_5 & \sqrt{3}F_8 + F_3 \end{array}$$

- For each SU(2) subgroup we can form the usual raising and lowering operators
- Any two of the three subgroups are enough to navigate through all the members of the multiplet
- Fundamental representation: A triplet



- Define group structure of the state by starting at one corner and using raising and lowering operators

$$\begin{aligned} (V_-)^{p+1}\phi_{max} &= 0 \\ (I_-)^{1+1}\phi_{max} &= 0 \\ \text{structure}: &\quad (p, q) \end{aligned}$$

- So quarks (u, d, s) have $p = 1,\ q = 0$ while antiquarks $(\overline{u}, \overline{d}, \overline{s})$ have $p = 0,\ q = 1$

Combining SU(3) states (2 quarks)

- Combining two SU(3) objects gives $3 \times 3 = 9$ possible states

$$
\begin{array}{cc}
uu & \\
\frac{1}{\sqrt{2}}(ud + du) & \frac{1}{\sqrt{2}}(ud - du) \\
dd & \\
\frac{1}{\sqrt{2}}(us + su) & \frac{1}{\sqrt{2}}(us - su) \\
ss & \\
\frac{1}{\sqrt{2}}(ds + sd) & \frac{1}{\sqrt{2}}(ds - sd) \\
\mathbf{6} & \overline{\mathbf{3}}
\end{array}
$$

$$3 \otimes 3 \quad = \quad 6 \oplus \overline{3}$$

- We know that the triplet is a $\overline{3}$ from its I_3 and Y:

Combining SU(3) states (a 3$^{\rm rd}$ quark)

- $3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \overline{3}) = 10_s \oplus 8_{M,S} \oplus 8_{M,A} \oplus 1$
- Start with the fully symmetric part of the **6**:

$$\begin{array}{cl} uuu & \text{3 such states} \\ \frac{1}{\sqrt{3}}(ddu + udd + dud) & \text{6 such states} \\ \frac{1}{\sqrt{6}}(dsu + uds + sud + sdu + dus + usd) & \text{1 such state} \end{array}$$

  Ten states that are fully symmetric

- Now, the mixed symmetry part of the **6**:

$$\frac{1}{\sqrt{6}}\left[(ud + du)u - 2uud\right] \quad \text{8 such states}$$

  Eight states like this

- Now on to the $\overline{3}$:

$$\frac{1}{\sqrt{6}}\left[(ud - du)s + (usd - dsu) + (du - ud)s\right] \quad \text{8 such states}$$

  Eight states like this

- Final state, totally antisymmetric

Combining SU(3) states ($q\overline{q}$)

- Start with $\pi^+ = u\,\overline{d}$
- Using:

$$I_-\left|\overline{u}\right\rangle = -\left|\overline{d}\right\rangle$$
$$I_-\left|\overline{d}\right\rangle = +\left|\overline{u}\right\rangle$$

We find:

$$\begin{aligned} I_-\left|u\overline{d}\right\rangle &= -\left|uu\right\rangle + \left|dd\right\rangle \\ &= \sqrt{2}\left|I = 1\ I_3 = 0\right\rangle \\ \pi^0 &= \frac{1}{\sqrt{2}}\left(\left|d\overline{d}\right\rangle - \left|u\overline{u}\right\rangle\right) \end{aligned}$$

Doing this again: $\pi^- = d\,\overline{u}$

- Now add strange quarks: 4 combinations

$u\overline{s}$	$d\overline{s}$	$\overline{u}s$	$\overline{d}s$
K^+	K^0	K^-	$\overline{K^0}$

- One missing combination:

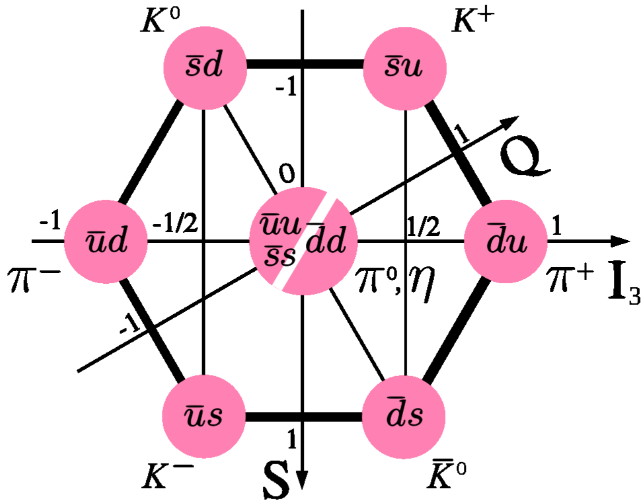
$$(d\overline{d} + u\overline{u} - 2s\overline{s})/\sqrt{6} \equiv \eta'$$

These 8 states are called an octet

- One additional independent combination: the singlet state

$$(u\overline{u} + d\overline{d} + s\overline{s})/\sqrt{6}$$

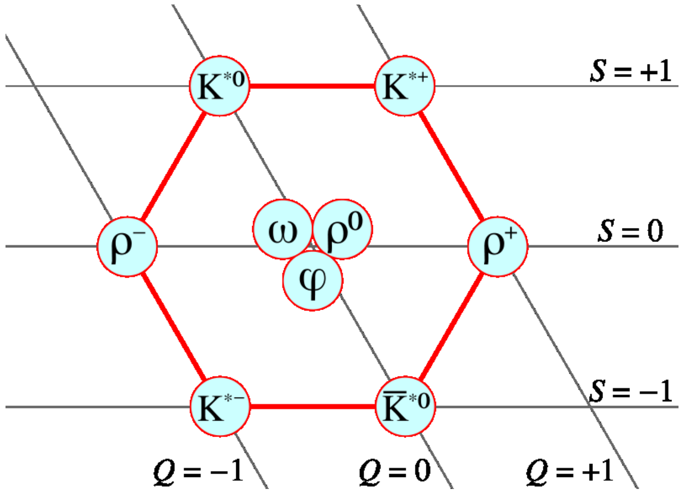
Pseudoscalar Mesons (0^-)



I	I_3	S	Meson	Combo	Decay	Mass (MeV)
1	1	0	π^+	$u\overline{d}$	$\mu^+\nu$	140
1	0	0	π^0	$\frac{1}{2}(d\overline{d} - u\overline{u})$	$\gamma\gamma$	135
1	-1	0	π^-	$d\overline{u}$	$\mu^-\overline{\nu}$	140
$\frac{1}{2}$	$\frac{1}{2}$	$+1$	K^+	$u\overline{s}$	$\mu^+\nu$	494
$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	K^0	$d\overline{s}$	$\pi^+\pi^-$	498
$\frac{1}{2}$	$\frac{1}{2}$	-1	K^-	$\overline{u}s$	$\mu^-\overline{\nu}$	494
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\overline{K^0}$	$\overline{d}s$	$\pi^+\pi^-$	498
0	0	0	η_8	$\frac{1}{\sqrt{6}}(d\overline{d} + u\overline{u} - 2s\overline{s})$	see below	
0	0	0	η_0	$\frac{1}{\sqrt{3}}(d\overline{d} + u\overline{u} + s\overline{s})$	see below	

- Mass of strange mesons larger than non-strange by 150 MeV
  - Strange quark has a larger mass than up and down
  - Leads to SU(3) breaking in H
- The η_8 and η_0 are degenerate if SU(3) were a perfect symmetry
  - Degenerate Pertubation Theory: The states can mix. Physical states are:
    * η: Mass=549 Decay:$\eta \to 2\gamma$
    * η': Mass=958 Decays:$\eta' \to \eta\pi\pi$ or $\gamma\gamma$

Vector Mesons (1^-)

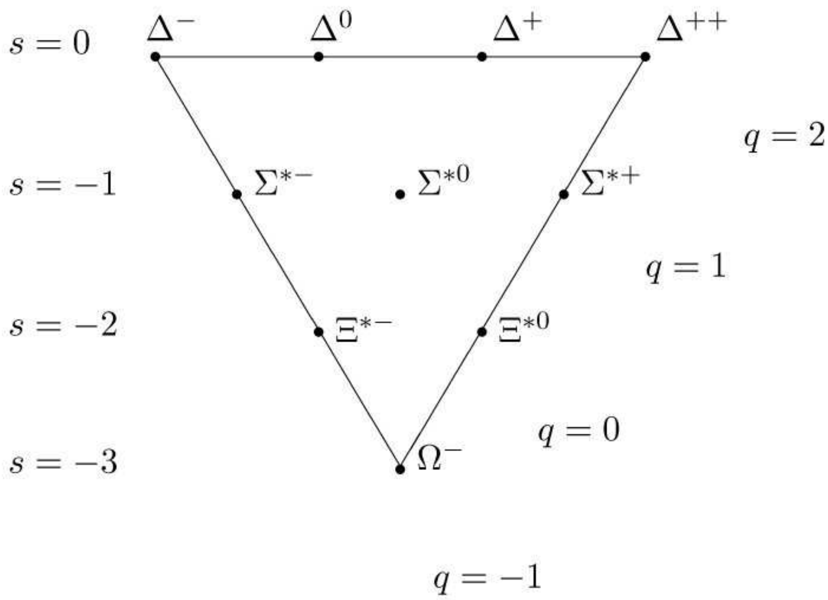


I	I_3	S	Meson	Combo	Decay	Mass (MeV)
1	1	0	ρ^+	$u\overline{d}$	$\pi^+\pi^0$	776
1	0	0	ρ^0	$\frac{1}{2}(d\overline{d}-u\overline{u})$	$\pi^+\pi^-$	776
1	-1	0	ρ^-	$d\overline{u}$	$\pi^-\pi^0$	776
$\frac{1}{2}$	$\frac{1}{2}$	$+1$	K^{*+}	$u\overline{s}$	$K\pi$	892
$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	K^{*0}	$d\overline{s}$	$K\pi$	892
$\frac{1}{2}$	$\frac{1}{2}$	-1	K^{*-}	$\overline{u}s$	$\overline{K}\pi$	892
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\overline{K^{*0}}$	$\overline{d}s$	$\overline{K}\pi$	892
0	0	0	ω	$\frac{1}{2}(u\mp d\overline{d})$	783	3π
0	0	0	ϕ	$s\overline{s}$	1019	$K\overline{K}$

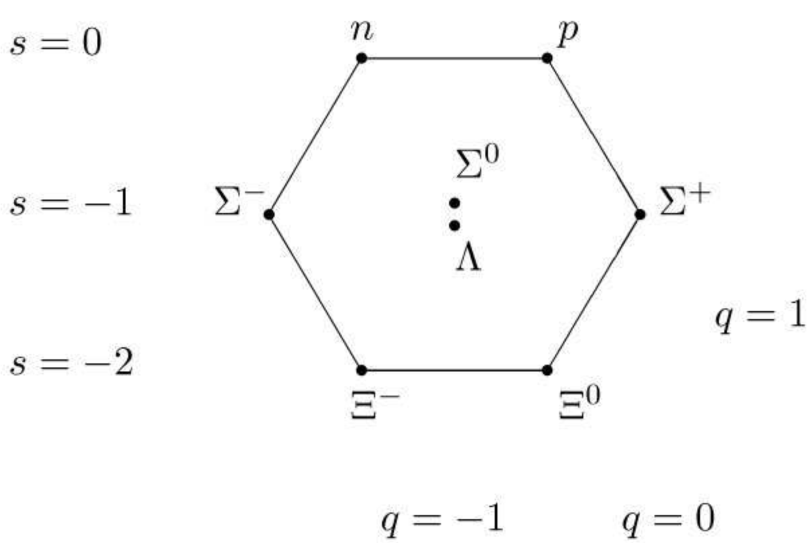
Last two states are given <u>after</u> the octet-singlet mixing. See the previous page for the SU(3) symmetric wave functions

- Unlike the pseudoscalars which decay weakly, the vectors can decay strongly
- The octet-nonet mixing is maximal in the case of the vector mesons
  - The ϕ is all $s\overline{s}$ while the ω is all $u\overline{u}$ and $d\overline{d}$

Baryon Decouplet ($\frac{3}{2}^-$)



Baryon Octet ($\frac{1}{2}^-$)



Comments on Antiparticles

- For mesons, particle and antiparticle are in the same multiplet
  - The multiplet is called “self-charge conjugate”
- For baryons, the antiparticles are in different multiplets
  - $\mathbf{10} \Rightarrow \overline{\mathbf{10}}$
  - $\mathbf{8} \Rightarrow \overline{\mathbf{8}}$

A Comment on Fermi Statistics: Why Color

- Imposition of Fermi Statistics on Baryon States
  - $\Delta^{++} = uu$, spin=$3/2$, s-wave: These are all symmetric under interchange
  - Need another degree of freedom to antisymmetrize (must haver at least 3 possible states, since we are antisymmetrizing 3 objects)
- We'll see the week after next, that the QCD Lagrangian is based on an $\mathrm{SU}(3)_{\mathrm{color}}$ interaction
  - Gluons are color octets
  - Observable hadrons are color singlets
  - The color singlet states are anti-symmetric under color exchange
    - $*$ This solves the Fermi statistics problem

SU(3) Breaking and Mass Relations

- In SU(3) symmetric world, all members of a multiplet should have the same mass
  - Value of mass depends on binding energy: cannot calculate this since perturbative non-relativistic calculations not possible for low energy QCD
- Several reasons why the physical masses of the hadrons in a multiple are different
  - Difference in quark masses
    - $m_d > m_u$ by a few MeV, m_s heavier by ~ 175 MeV
  - Coulomb energy diffference associated with the electrical energy between pairs of quarks
    - Of order e^2/R_0. With $R_0 \sim 0.8$ fm, this ~ 2 MeV
  - Magnetic energy differences associated with the magnetic moments of the quarks (hyperfine interaction)
    - Standard EM hyperfine splitting:

$$\Delta E = \vec{\mu} \cdot \vec{B} = \frac{2}{3} \vec{\mu}_i \cdot \vec{\mu}_j |\psi(0)|^2 = \frac{2\pi}{3} \frac{\alpha}{m_i m_j} \vec{\sigma}_i \cdot \vec{\sigma}_j |\psi(0)|^2 \quad \text{(a few MeV)}$$

    - Color hyperfine splitting has same basic structure except α is replaced with α_s and the numerical factor in front is different (due to strong interaction color factors)

$$-\alpha \to \begin{cases} -\frac{4}{3}\alpha_s & \text{for } q\overline{q} \\ -\frac{2}{3}\alpha_s & \text{for } qqq \end{cases}$$